

# Minimal Length, Measurability and Gravity

A.E.Shalyt-Margolin <sup>1</sup>

*Research Institute for Nuclear Problems, 11 Bobruiskaya str., Minsk  
220040, Belarus*

PACS: 03.65, 05.20

Keywords: minimal length, measurability, gravity

## Abstract

The present work is a continuation of the previous papers written by the author on the subject. In terms of the measurability (or measurable quantities) notion introduced in a minimal length theory, first the consideration is given to a quantum theory in the momentum representation. The same terms are used to consider the Markov gravity model that here illustrates the general approach to studies of gravity in terms of measurable quantities.

This paper is dedicated to the 75th Anniversary of Professor Vladimir Grigor'evich Baryshevsky.

## 1 Introduction: Measurable and Nonmeasurable Quantities

This work is a direct continuation of my previous papers [1, 2] and is interlaced with these publications at some points. As shown in [1], provided the theory involves the minimal length  $l_{min}$  as a *minimal measurement unit* for the quantities having the dimensions of length, this theory must not have infinitesimal spatial-temporal quantities  $dx_\mu$  because the latter lead to the infinitely small length  $ds$  [3]

$$ds^2 = g_{\mu\nu} dx_\mu dx_\nu \quad (1)$$

---

<sup>1</sup>E-mail:  
Tel.: +375-17-292-60-34

a.shalyt@mail.ru;

alexm@hep.by;

that is inexistent because of  $l_{min}$ .

Of course, in this case only *measurable* quantities are meant. As a mathematical notion, the quantity  $ds$  is naturally existent but, due to the involvement of  $l_{min}$ , it is immeasurable.

However it is well known that at high energies (on the order of the quantum gravity energies) the minimal length  $l_{min}$  to which the indicated energies are “sensitive”, as distinct from the low ones, should inevitably become apparent in the theory. But if  $l_{min}$  is really present, it must be present at all the “Energy Levels” of the theory, low energies including. And this, in addition to the above arguments, points to the fact that the mathematical formalism of the theory should not involve any infinitesimal spatial-temporal quantities. Besides, some new parameters become involved, which are dependent on  $l_{min}$  [4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

What are the parameters of interest in the case under study? It is obvious that, as the quantum-gravitational effects will be revealed at very small (possibly Planck) scales, these parameters should be dependent on some limiting values, e.g.,  $l_P \propto l_{min}$  and hence Planck energy  $E_P$ .

*This means that in a high-energy gravitation theory the energy-dependent or, what is the same, measuring scales-dependent parameters should be necessarily introduced.*

But, on the other hand, these parameters could hardly disappear totally at low energies, *i.e.*, for General Relativity (GR) too. However, since the well-known canonical (and in essence the classical) statement of GR has no such parameters [3], the inference is as follows: their influence at low energies is so small that it may be disregarded at the modern stage in evolution of the theory and of the experiment.

Still this does not imply that they should be ignored in future evolution of the theory, especially on going to its high-energy limit.

But at the present time, the mathematical apparatus of both special and general relativity theories (and of a quantum theory as well) is based on the concept of continuity and on analysis of infinitesimal spatial-temporal quantities. This is a corner stone for the Minkowski space geometry (MS) and also for the Riemannian geometry (RG) [3].

However, this approach involves a problem when we proceed to a quantum description of nature. Even at a level of the heuristic understanding, it

is clear that, as measuring procedures in a quantum theory are fundamental, the description with the use of infinitesimal quantities is problematic because in its character the measuring procedure is discrete.

At a level of the mathematical formalism and physical principles, incompatibility of both the Minkowski space geometry and Riemannian geometry with the uncertainty principle is expected in any “format”, in relativistic and nonrelativistic cases. This problem is considered in greater detail in the following sections of this work.

Thus, if the matter concerns the measurable quantities only, the Quantum Theory (QT) and Gravity formalism should be changed: at least, a new formalism should not involve the infinitesimal spatial-temporal quantities  $dx_\mu$ . Naturally, because of the involved  $l_{min}$  (initially assuming that  $l_{min} \propto l_P$ ) new theories should involve new parameters associated with  $l_{min}$ . Presently, such parameters are inexplicitly involved (for example,  $E/E_P$  in a quantum gravity phenomenology [4]).

But there is no need to discard the modern formalism of QT and Gravity, since it is clear that at low energies it offers an excellent approximation, experimentally supported to a high accuracy (see [5]). However, proceeding from the above, a change-over to high energies is impossible as, by author’s opinion, this formalism is used in an effort to combine uncombinable things.

This work makes the arguments of [1, 2] more forcible on the one hand, and presents a study of the additional parameters associated with the involvement of  $l_{min}$ , in terms of which one can develop a new formalism for a quantum theory and for gravity at all the scales energies too, on the other hand.

One of the key problems of the modern fundamental physics (Quantum Theory (QT) and Gravity (GR)) is framing of a correct theory associated with the ultraviolet region, *i.e.*, the region of the highest (apparently Planck) energies approaching those of the Big Bang.

However, it is well known that at high energies (on the order of the quantum gravity energies) the minimal length  $l_{min}$  to which the indicated energies are “sensitive”, as distinct from the low ones, should inevitably become apparent in the theory. But if  $l_{min}$  is really present, it must be present at all the “Energy Levels” of the theory, low energies including.

What follows from the existence of the minimal length  $l_{min}$ ? When the

minimal length is involved, any nonzero measurable quantity having the dimensions of length should be a multiple of  $l_{min}$ . Otherwise, its measurement with the use of  $l_{min}$  would result in the measurable quantity  $\varsigma$ , so that  $\varsigma < l_{min}$ , and this is impossible.

This suggests that the spatial-temporal quantities  $dx_\mu$  are nonmeasurable quantities because the latter lead to the infinitely small nonmeasurable quantity length  $ds$  Equation (1).

Of course, as a mathematical notion, the quantities  $dx_\mu, ds$  are naturally existent but one should realize that there is no way to express them in terms of the minimal possible measuring unit  $l_{min}$ .

So, trying to frame a theory (QT and GR) correct at all the energy levels using only the measurable quantities, one should realize that then the mathematical formalism of the theory should not involve any infinitesimal spatial-temporal quantities. Besides, proceeding from the acknowledged results associated with the Planck scales physics [4, 5, 6, 7, 8, 9, 10, 11, 12, 13], one can infer that certain new parameters dependent on  $l_{min}$  should be involved.

What are the parameters of interest in the case under study? It is obvious that, as the quantum-gravitational effects will be revealed at very small (possibly Planck) scales, these parameters should be dependent on some limiting values, e.g.,  $l_P \propto l_{min}$  and hence Planck's energy  $E_P$ .

This means that in high-energy QT and GR the energy-or, what is the same, measuring scales-dependent parameters should be necessarily introduced.

But, on the other hand, these parameters could hardly disappear totally at low energies both in QT and in GR.

But, provided  $l_{min}$  exists, it must be involved at all the energy levels, both high and low.

The fact that  $l_{min}$  is omitted in the formulation of low-energy QT and GR and the theories give perfect results leads to two different inferences:

1. The influence of the above-mentioned new parameters associated with  $l_{min}$  in low-energy QT and GR is so small that it may be disregarded at the modern stage in evolution of the theory and of the experiment.

2. The modern mathematical apparatus of conventional QT and GR has been derived in terms of the infinitesimal spatial-temporal quantities  $dx_\mu$  which, as noted above, are nonmeasurable quantities in the formalism of  $l_{min}$ .

## 2 Main Motivation

In this Section the principal assumptions are introduced which have been implicitly used previously in [1] and especially in [2].

It is well known that in a quantum study the key role is played by the measuring procedure, its fundamental principle being the Heisenberg Uncertainty Principle (HUP) [14, 15]:

$$\Delta x \geq \frac{\hbar}{\Delta p} \quad (2)$$

(Note that the normalization  $\Delta x \Delta p \geq \hbar$  is used rather than  $\Delta x \Delta p \geq \hbar/2$ .)

Now we can proceed to the following quite natural suppositions.

**Supposition 1.** Any small variation (increment)  $\tilde{\Delta}x_\mu$  of any spatial coordinate  $x_\mu$  of the arbitrary point  $x_\mu, \mu = 1, \dots, 3$  in some space-time system  $R$  may be realized in the form of the uncertainty (standard deviation)  $\Delta x_\mu$  when this coordinate is measured within the scope of Heisenberg's Uncertainty Principle (HUP)

$$\tilde{\Delta}x_\mu = \Delta x_\mu, \Delta x_\mu \simeq \frac{\hbar}{\Delta p_\mu}, \mu = 1, 2, 3 \quad (3)$$

for some  $\Delta p_\mu \neq 0$ .

Similarly, for  $\mu = 0$  for pair “time-energy”  $(t, E)$ , the any small variation (increment) value of time  $\tilde{\Delta}x_0 = \tilde{\Delta}t_0$  may be realized in the form of the uncertainty (standard deviation)  $\Delta x_0 = \Delta t$  and then

$$\tilde{\Delta}t = \Delta t, \Delta t \simeq \frac{\hbar}{\Delta E} \quad (4)$$

for some  $\Delta E \neq 0$ .

Here HUP is given for the nonrelativistic case. In the relativistic case HUP has the distinctive features [16] which, however, are of no significance for the general formulation of Supposition 1, being associated only with particular alterations in the right-hand side of the second relation Equation (3) as shown later.

It is clear that at low energies  $E \ll E_P$  (momentums  $P \ll P_{pl}$ ) Supposition 1 sets a lower bound for the variations (increments)  $\tilde{\Delta}x_\mu$  of any space-time coordinate  $x_\mu$ .

At high energies  $E$  (momentums  $P$ ) this is not the case if  $E$  ( $P$ ) have no upper limit. But, according to the modern knowledge,  $E$  ( $P$ ) are bounded by some maximal quantities  $E_{max}$ , ( $P_{max}$ )

$$E \leq E_{max}, P \leq P_{max}, \quad (5)$$

where in general  $E_{max}, P_{max}$  may be on the order of Planck quantities  $E_{max} \propto E_P, P_{max} \propto P_{pl}$  and also may be the trans-Planck's quantities.

In any case the quantities  $P_{max}$  and  $E_{max}$  lead to the introduction of the minimal length  $l_{min}$  and of the minimal time  $t_{min}$ .

With this point of view, even at the ultimate (Planck) energies a minimal “detected” (*i.e.*, measurable) space-time volume is, within the known constants, restricted to

$$V_{min} \propto l_P^4. \quad (6)$$

Consequently, “detectability” of the infinitesimal space-time volume

$$V_{dx_\mu} = (dx_\mu)^4 \quad (7)$$

is impossible as this necessitates going to infinitely high energies

$$E \rightarrow \infty. \quad (8)$$

Because of this, it is natural to complete Supposition 1 with Supposition 2.

**Supposition 2.** There is the minimal length  $l_{min}$  as a minimal measurement unit for all quantities having the dimension of length, whereas the minimal time  $t_{min} = l_{min}/c$  as a minimal measurement unit for all quantities having the dimension of time, where  $c$  is the speed of light.

$l_{min}$  and  $t_{min}$  are naturally introduced as  $\Delta x_\mu, \mu = 1, 2, 3$  and  $\Delta t$  in Equations (3) and (4) for  $\Delta p_\mu = P_{max}$  and  $\Delta E = E_{max}$ .

For definiteness, we consider that  $E_{max}$  and  $P_{max}$  are the quantities on the order of the Planck quantities, then  $l_{min}$  and  $t_{min}$  are also on the order of Planck quantities  $l_{min} \propto l_P$ ,  $t_{min} \propto t_P$ .

Suppositions 1 and 2 are quite natural in the sense that there are no physical principles with which these suppositions are inconsistent.

### 3 Minimal Length and Measurability

In this Section particularly the results from Subsection 3.1 of [2] are used. Now from the start we assume that the theory involves the minimal length  $l_{min}$  as a minimal measurement unit for all quantities having the dimension of length.

Then it is convenient to begin our study not with HUP Equation (2) but with its widely known high-energy generalization—the Generalized Uncertainty Principle (GUP) that naturally leads to the minimal length  $l_{min}$  [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]:

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' l_P^2 \frac{\Delta p}{\hbar}. \quad (9)$$

Here  $\alpha'$  is the model-dependent dimensionless numerical factor and  $l_P$  is the Planckian length.

Note also that initially GUP Equation (9) was derived within a string theory [17, 18, 19, 20] and, subsequently, in a series of works far from this theory [21, 22, 23, 24, 25, 26, 27] it has been demonstrated that on going to high (Planck) energies in the right-hand side of HUP Equation (2) an additional “high-energy” term  $\propto l_P^2 \frac{\Delta p}{\hbar}$  appears. Of particular interest is the work [21], where by means of a simple gedanken experiment it has been demonstrated that with regard to the gravitational interaction Equation (9) is the case.

As Equation (9) is a quadratic inequality, then it naturally leads to the minimal length  $l_{min} = \xi l_P = 2\sqrt{\alpha'} l_P$ .

This means that the theory for the quantities with a particular dimension has a minimal measurement unit. At least, all the quantities with such a

dimension should be “quantized”, *i.e.*, be measured by an integer number of these minimal units of measurement.

Specifically, if  $l_{min}$ —minimal unit of length, then for any length  $L$  we have the “Integrality Condition” (IC)

$$L = N_L l_{min}, \quad (10)$$

where  $N_L > 0$  is an integer number.

What are the consequences for GUP Equation (9) and HUP Equation (2)?

Assuming that HUP is to a high accuracy derived from GUP on going to low energies or that HUP is a special case of GUP at low values of the momentum, we have

$$(GUP, \Delta p \rightarrow 0) = (HUP). \quad (11)$$

By the language of  $N_L$  from Equations (10) and (11) is nothing else but a change-over to the following:

$$(N_{\Delta x} \approx 1) \rightarrow (N_{\Delta x} \gg 1). \quad (12)$$

The assumed equalities in Equations (2) and (9) may be conveniently rewritten in terms of  $l_{min}$  with the use of the deformation parameter  $\alpha_a$ . This parameter has been introduced earlier in the papers [30, 31, 32, 33, 34, 35, 36, 37] as a deformation parameter on going from the canonical quantum mechanics to the quantum mechanics at Planck’s scales (early Universe) that is considered to be the quantum mechanics with the minimal length (QMML):

$$\alpha_a = l_{min}^2/a^2, \quad (13)$$

where  $a$  is the measuring scale.

### Definition 1

*Deformation is understood as an extension of a particular theory by inclusion of one or several additional parameters in such a way that the initial theory appears in the limiting transition [38].*



Then with the equality ( $\Delta p \Delta x = \hbar$ ) Equation (9) is of the form

$$\Delta x = \frac{\hbar}{\Delta p} + \frac{\alpha_{\Delta x}}{4} \Delta x. \quad (14)$$

In this case due to Equations (10), (12) and (14) takes the following form:

$$N_{\Delta x} l_{min} = \frac{\hbar}{\Delta p} + \frac{1}{4N_{\Delta x}} l_{min} \quad (15)$$

or

$$(N_{\Delta x} - \frac{1}{4N_{\Delta x}}) l_{min} = \frac{\hbar}{\Delta p}. \quad (16)$$

That is

$$\Delta p = \frac{\hbar}{(N_{\Delta x} - \frac{1}{4N_{\Delta x}}) l_{min}}. \quad (17)$$

From Equations (15)–(17) it is clear that HUP Equation (2) in the case of the equality appears to a high accuracy in the limit  $N_{\Delta x} \gg 1$  in conformity with Equation (12).

It is easily seen that the parameter  $\alpha_a$  from Equation (13) is discrete as it is nothing else but

$$\alpha_a = l_{min}^2 / a^2 = \frac{l_{min}^2}{N_a^2 l_{min}^2} = \frac{1}{N_a^2}. \quad (18)$$

At the same time, from Equation (18) it is evident that  $\alpha_a$  is irregularly discrete.

It is clear that from Equation (17) at low energies ( $N_{\Delta x} \gg 1$ ), up to a constant

$$\frac{\hbar^2}{l_{min}^2} = \frac{\hbar c^3}{4\alpha' G} \quad (19)$$

we have

$$\alpha_{\Delta x} = (\Delta p)^2. \quad (20)$$

But all the above computations are associated with the nonrelativistic case. As known, in the relativistic case, when the total energy of a particle with the mass  $m$  and with the momentum  $p$  equals [39]:

$$E = \sqrt{p^2 c^2 + m^2 c^4}, \quad (21)$$

a minimal value for  $\Delta x$  takes the form [16]:

$$\Delta x \approx \frac{c\hbar}{E}. \quad (22)$$

And in the ultrarelativistic case

$$E \approx pc \quad (23)$$

this means simply that

$$\Delta x \approx \frac{\hbar}{p}. \quad (24)$$

Provided the minimal length  $l_{min}$  is involved and considering the “Integrality Condition” (IC) Equation (10), in the general case for Equation (22) at the energies considerably lower than the Planck energies  $E \ll E_P$  we obtain the following:

$$\begin{aligned} \Delta x = N_{\Delta x} l_{min} &\approx \frac{c\hbar}{E}, \\ &\text{or} \\ E &\approx \frac{c\hbar}{N_{\Delta x}}. \end{aligned} \quad (25)$$

Similarly, at the same energy scale in the ultrarelativistic case we have

$$p \approx \hbar/N_{\Delta x}. \quad (26)$$

Next under Supposition 2, we assume that there is a minimal measuring unit of time

$$t_{min} = l_{min}/v_{max} = l_{min}/c. \quad (27)$$

Then the foregoing Equations (2)–(16) are rewritten by substitution as follows:

$$\Delta x \rightarrow \Delta t; \Delta p \rightarrow \Delta E; l_{min} \rightarrow t_{min}; N_L \rightarrow N_{t=L/c} \quad (28)$$

Specifically, Equation (16) takes the form

$$(N_{\Delta t} - \frac{1}{4N_{\Delta t}})t_{min} = \frac{\hbar}{\Delta E}. \quad (29)$$

And similar to Equation (10), we get the “Integrality Condition” (IC) for any time  $t$ :

$$t \equiv t(N_t) = N_t t_{min}, \quad (30)$$

for certain an integer  $|N_t| \geq 0$ .

According to Equation (29), let us define the corresponding energy  $E$

$$E \equiv E(N_t) = \frac{\hbar}{|N_t - \frac{1}{4N_t}| t_{min}}. \quad (31)$$

Note that at low energies  $E \ll E_P$ , that is for  $|N_t| \gg 1$ , the formula Equation (31) naturally takes the following form:

$$E \equiv E(N_t) = \frac{\hbar}{|N_t| t_{min}} = \frac{\hbar}{|t(N_t)|}. \quad (32)$$

**Definition 2 (Measurability)**

- (1) *Let us define the quantity having the dimensions of length  $L$  or time  $t$  measurable, when it satisfies the relation Equation (10) (and respectively Equation (30)).*
- (2) *Let us define any physical quantity measurable, when its value is consistent with points (1) of this Definition.*

Thus, infinitesimal changes in length (and hence in time) are impossible (to that indicated in Section 1) and any such changes are dependent on the existing energies.

In particular, a minimal possible measurable change of length is  $l_{min}$ . It corresponds to some maximal value of the energy  $E_{max}$  or momentum  $P_{max}$ , If  $l_{min} \propto l_P$ , then  $E_{max} \propto E_P, P_{max} \propto P_{Pl}$ , where  $P_{max} \propto P_{Pl}$ , where  $P_{Pl}$  is where the Planck momentum. Then denoting in nonrelativistic case with  $\Delta_p(w)$  a minimal measurable change every spatial coordinate  $w$  corresponding to the energy  $E$  we obtain

$$\Delta_{P_{max}}(w) = \Delta_{E_{max}}(w) = l_{min}. \quad (33)$$

Evidently, for lower energies (momentums) the corresponding values of  $\Delta_p(w)$  are higher and, as the quantities having the dimensions of length are quantized Equation (10), for  $p \equiv p(N_p) < p_{max}$ ,  $\Delta_p(w)$  is transformed to

$$|\Delta_{p(N_p)}(w)| = |N_p| l_{min}. \quad (34)$$

where  $|N_p| > 1$  is an integer number so that we have

$$|N_p - \frac{1}{4N_p}|l_{min} = \frac{\hbar}{|p(N_p)|}. \quad (35)$$

In the relativistic case the Equation (33) holds, whereas Equations (34) and (35) for  $E \equiv E(N_E) < E_{max}$  are replaced by

$$|\Delta_{E(N_E)}(w)| = |N_E|l_{min}, \quad (36)$$

where  $|N_E| > 1$  is an integer.

Next we assume that at high energies  $E \propto E_P$  there is a possibility only for the nonrelativistic case or ultrarelativistic case.

Then for the ultrarelativistic case, with regard to Equations (23)–(29), Formula (35) takes the form

$$|N_E - \frac{1}{4N_E}|l_{min} = \frac{\hbar c}{E(N_E)} = \frac{\hbar}{|p(N_p)|}, \quad (37)$$

where  $N_E = N_p$ .

In the relativistic case at low energies we have

$$E \ll E_{max} \propto E_P. \quad (38)$$

In accordance with Equations (21) and (22) and Formula (34) is of the form

$$|\Delta_{E(N_E)}(w)| = |N_E|l_{min} = \frac{\hbar c}{E(N_E)}, |N_E| \gg 1 - integer. \quad (39)$$

In the nonrelativistic case at low energies Equation (38) due to Equation (35) we get

$$|\Delta_{p(N_p)}(w)| = |N_p|l_{min} = \frac{\hbar}{|p(N_p)|}, |N_p| \gg 1 - integer. \quad (40)$$

In a similar way for the time coordinate  $t$ , by virtue of Equations (30)–(32), at the same conditions we have similar Equations (33)–(35)

$$\Delta_{E_{max}}(t) = t_{min}. \quad (41)$$

For  $E \equiv E(N_t) < E_{max}$

$$|\Delta_{E(N_t)}(t)| = |N_t|t_{min}, \quad (42)$$

where  $|N_E| > 1$  is an integer, so that we obtain

$$|N_t - \frac{1}{4N_t}|t_{min} = \frac{\hbar c}{E(N_t)}. \quad (43)$$

In the relativistic case at low energies

$$E \ll E_{max} \propto E_P, \quad (44)$$

in accordance with Equations (21) and (22), Equation (34) takes the form

$$|\Delta_{E(N_t)}(w)| = |N_t|l_{min} = \frac{\hbar c}{E(N_t)}, |N_t| \gg 1 - integer. \quad (45)$$

**Remark 1.**

**1.1.** It should be noted that the lattice is usually understood as a uniform discrete structure with one and the same constant parameter  $a$  (lattice pitch). But in this case we have a nonuniform discrete structure (lattice in its nature), where the analogous parameter is variable, is a multiple of  $l_{min}$ , *i.e.*,  $a = N_a l_{min}$ , and also is dependent on the energies. Only in the limit of high (Planck's) energies we get a (nearly) uniform lattice with (nearly) constant pitch  $a \approx l_{min}$  or  $a = \kappa l_{min}$  where  $\kappa$  is on the order of 1.

**1.2.** Obviously, when  $l_{min}$  is involved, the foregoing formulas for the momenta  $p(N_p)$  and for the energies  $E(N_E), E(N_t)$  may certainly give the highly accurate result that is close to the experimental one only at the verified low energies:  $|N_p| \gg 1, |N_E| \gg 1, |N_t| \gg 1$ . In the case of high energies  $E \propto E_{max} \propto E_P$  or, what is the same  $|N_p| \rightarrow 1, |N_E| \rightarrow 1, |N_t| \rightarrow 1$ , we have a certain, experimentally unverified, model with a correct low-energy limit.

**1.3.** It should be noted that dispersion relations Equation (21) are valid

only at low energies  $E \ll E_P$ . In the last few years in a series of works [40, 41, 42, 43, 44] it has been demonstrated that within the scope of GUP the high-energy generalization of Equation (21)—Modified Dispersion Relations (MDRs)—is valid.

Specifically, in its most general form the Modified Dispersion Relation (Formula (9) in [44]) may be given as follows:

$$p^2 = f(E, m; l_p) \simeq E^2 - \mu^2 + \alpha_1 l_p E^3 + \alpha_2 l_p^2 E^4 + O(l_p^3 E^5), \quad (46)$$

where in the notation of [44] the fundamental constants are  $c = \hbar = k_B = 1$ ,  $f$  is the function that gives the exact dispersion relation, and in the right-hand side the applicability of the Taylor-series expansion for  $E \ll 1/l_P$  is assumed. The coefficients  $\alpha_i$  can take different values in different quantum-gravity proposals.  $m$  is the rest energy of a particle, and the mass parameter  $\mu$  in the right-hand side is directly related to the rest energy but  $\mu \neq m$  if not all the coefficients  $\alpha_i$  are vanishing.

The general case of (MDRs) Equation (46) in terms of the considerations given in this section is yet beyond the scope of this paper and necessitates further studies of the transition from low  $E \ll E_P$  to high  $E \approx E_P$  energies.

For now it is assumed that at low energies Equation (21) is valid to within a high accuracy, whereas at high energies, *i.e.*, for  $|N_p| \rightarrow 1, |N_E| \rightarrow 1, |N_t| \rightarrow 1$ , Equation (21) should be replaced by Equation (46). Besides, it is important that in this paper, as distinct from [40, 41, 42, 43, 44], the author uses the simplest (earlier) variant of GUP [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27], involving a minimal length but not a minimal momentum.

Also note that references [40, 41, 42, 43, 44] give not nearly so complete a list of the publications devoted to GUP (and, in particular, MDR)—a very complete and interesting survey may be found in [41].

**1.4.** The papers [1, 2] point to the fact that the resolved discrete theory is very close to the initial continuous one ( $l_{min} = 0$ ) at low energies  $E \ll E_P$ , *i.e.*, at  $|N_p| \gg 1, |N_E| \gg 1$ .

In what follows all the considerations are given in terms of “measurable quantities” in the sense of Definition 2 given in this Section. Specifically, in Section 5 these terms are used to consider the Momentum Representation

for a quantum theory.

## 4 Space-Time Lattice of Measurable Quantities and Dual Lattice

So, provided the minimal length  $l_{min}$  exists, two lattices are naturally arising.

**I.** Lattice of the space-time variation— $Lat_{S-T}$  representing, to within the known multiplicative constants, the sets of nonzero integers  $N_w \neq 0$  and  $N_t \neq 0$  in the corresponding formulas from the set Equations (34) and (45) for each of the three space variables  $w \doteq x; y; z$  and the time variable  $t$

$$Lat_{S-T} \doteq (N_w, N_t). \quad (47)$$

Which restrictions should be initially imposed on these sets of nonzero integers?

It is clear that in every such set all the integers  $(N_w, N_t)$  should be sufficiently “close”, because otherwise, for one and the same space-time point, variations in the values of its different coordinates are associated with principally different values of the energy  $E$  which are “far” from each other.

Note that the words “close” and “far” will be elucidated further in this text.

Thus, at the admittedly low energies (Low Energies)  $E \ll E_{max} \propto E_P$  the low-energy part (sublattice)  $Lat_{S-T}[LE]$  of  $Lat_{S-T}$  is as follows:

$$Lat_{S-T}[LE] = (N_w, N_t) \equiv (|N_x| \gg 1, |N_y| \gg 1, |N_z| \gg 1, |N_t| \gg 1). \quad (48)$$

At high energies (High Energies)  $E \rightarrow E_{max} \propto E_P$  we, on the contrary, have the sublattice  $Lat_{S-T}[HE]$  of  $Lat_{S-T}$

$$Lat_{S-T}[HE] = (N_w, N_t) \equiv (|N_x| \approx 1, |N_y| \approx 1, |N_z| \approx 1, |N_t| \approx 1). \quad (49)$$

**II.** Next let us define the lattice momenta-energies variation  $Lat_{P-E}$  as a set to obtain  $(p_x(N_{x,p}), p_y(N_{y,p}), p_z(N_{z,p}), E(N_t))$  in the nonrelativistic and ultrarelativistic cases for all energies, and as a set to obtain

$(E_x(N_{x,E}), E_y(N_{y,E}), E_z(N_{z,E}), E(N_t))$  in the relativistic (but not ultrarelativistic) case for low energies  $E \ll E_P$ , where all the components of the above sets conform to the space coordinates  $(x, y, z)$  and time coordinate  $t$  and are given by the corresponding Formulas (33)–(45) from the previous Section.

Note that, because of the suggestion made after formula Equation (38) in the previous Section, we can state that the foregoing sets exhaust all the collections of momentums and energies possible for the lattice  $Lat_{S-T}$ . From this it is inferred that, in analogy with point I of this Section, within the known multiplicative constants, we have

$$Lat_{P-E} \doteq (\frac{1}{N_w - \frac{1}{1/4N_w}}, \frac{1}{N_t - \frac{1}{1/4N_t}}), \quad (50)$$

where  $N_w \neq 0, N_t \neq 0$  are integer numbers from Equation (47). Similar to Equation (48), we obtain the low-energy (Low Energy) part or the sublattice  $Lat_{P-E}[LE]$  of  $Lat_{P-E}$

$$Lat_{P-E}[LE] \approx (\frac{1}{N_w}, \frac{1}{N_t}), |N_w| \gg 1, |N_t| \gg 1. \quad (51)$$

In accordance with Equation (49), the high-energy (High Energy) part (sublattice)  $Lat_{P-E}[HE]$  of  $Lat_{P-E}$  takes the form

$$Lat_{P-E}[HE] \approx (\frac{1}{N_w - \frac{1}{1/4N_w}}, \frac{1}{N_t - \frac{1}{1/4N_t}}), |N_w| \rightarrow 1, |N_t| \rightarrow 1. \quad (52)$$

Considering **Remark 1** from the previous Section, it should be noted that, as currently the low energies  $E \ll E_{max} \propto E_P$  are verified by numerous experiments and thoroughly studied, the sublattice  $Lat_{P-E}[LE]$  Equation (51) is correctly defined and rigorously determined by the sublattice  $Lat_{S-T}[LE]$  Equation (48).

However, at high energies  $E \rightarrow E_{max} \propto E_P$  we can not be so confident the sublattice  $Lat_{P-E}[HE]$  may be defined more exactly.

Specifically,  $\alpha_a$  is obviously a small parameter. And, as demonstrated in [45, 46], in the case of GUP we have the following:

$$[\vec{x}, \vec{p}] = i\hbar(1 + a_1\alpha + a_2\alpha^2 + \dots). \quad (53)$$



But, according to Equation (18),  $|1/N_a| = \sqrt{\alpha_a}$ , then, due to Equation (53), the denominators in the right-hand side of Equation (52) may be also varied by adding the terms  $\propto 1/N_w^2, \propto 1/N_w^3, \dots, \propto 1/N_t^2, \propto 1/N_t^3, \dots$ , that is liable to influence the final result for  $|N_w| \rightarrow 1, |N_t| \rightarrow 1$ . The notions “close” and “far” for  $Lat_{P-E}$  will be completely determined by the dual lattice  $Lat_{S-T}[LE]$  and by Formulas (34) and (45).

It is important to note the following.

In the low-energy sublattice  $Lat_{P-E}[LE]$  all elements are varying very smoothly enabling the approximation of a continuous theory.

## 5 Measurable Quantities and Momentum Representation

For convenience, we denote the minimal length  $l_{min} \neq 0$  by  $\ell$ .

Let us consider the above calculations using the formalism of the well-known work [28]. Then GUP (Section 3.2 in [28]) has the following form:

$$[\mathbf{x}, \mathbf{p}] = i\hbar(1 + \beta \mathbf{p}^2), \quad (54)$$

where ( $\beta > 0$ ) and

$$\beta = \frac{\ell^2}{\hbar^2}. \quad (55)$$

In the form of Section 3 in the present work, Formula (7) from [28]

$$\Delta x \Delta p \geq \hbar(1 + \beta(\Delta p)^2 + \beta \langle \mathbf{p} \rangle^2) \quad (56)$$

with regard to Equations (10), (15), (17) and (55) may be written as

$$\frac{\hbar N_{\Delta x}}{(N_{\Delta x} - \frac{1}{4N_{\Delta x}})} \geq \hbar(1 + \frac{1}{(N_{\Delta x} - \frac{1}{4N_{\Delta x}})^2} + \frac{\ell^2}{\hbar^2} \langle \mathbf{p} \rangle^2). \quad (57)$$

In the equality case this results in the following expression:

$$\frac{-\hbar^2(12N_{\Delta x}^2 + 1)}{(4N_{\Delta x}^2 - 1)^2 \ell^2} = \frac{-\hbar^2}{\ell^2} (3 + \frac{4}{(4N_{\Delta x}^2 - 1)^2}) = \langle \mathbf{p} \rangle^2. \quad (58)$$

In this way at low energies  $E \ll E_P$ , *i.e.*, at  $|N_{\Delta x}| \gg 1$ ,  $\langle \mathbf{p} \rangle^2$  is varying practically continuously.  
Next, hereinafter we use the Formula (35) with the replacement of  $l_{min} = \ell$ , *i.e.*, we have  
 $N_{\Delta x} = N_p$  and

$$|p_N| = \frac{\hbar}{|N_p - \frac{1}{4N_p}| \ell}. \quad (59)$$

We can write

$$i\hbar(1 + \beta p^2) = i\hbar(1 + \frac{\ell^2}{\hbar^2} \frac{\hbar^2}{(N_p - \frac{1}{4N_p})^2 \ell^2}) = i\hbar(1 + \frac{1}{(N_p - \frac{1}{4N_p})^2}). \quad (60)$$

Let us introduce the following symbols:

$$\begin{aligned} \Delta_p p_N = p_N - p_{N+1}; \Delta_p^{-1} \psi(p_N) &= \frac{\psi(p_N) - \psi(p_{N+1})}{p_N - p_{N+1}} = \\ &= \frac{\psi(p_{N+1} + \Delta_p p_N) - \psi(p_{N+1})}{\Delta_p p_N}. \end{aligned} \quad (61)$$

Then we suppose that only in the classical dynamics variations of momenta (energies) have no lower bounds and we have  $dp$ . At the same time, in a quantum dynamics, due to the limited spatial domains, these variations have both upper and lower bounds.

In this case, as distinct from [28], in the theory there is a minimum variation of the momentum  $\Delta p_{min}$  that within the scope of the measurability (Definition 2 in Section 3) takes the form

$$\Delta p_{min} \equiv p = \frac{\hbar}{\ell} \frac{1}{(\mathbf{N} - \frac{1}{4\mathbf{N}})} \approx \frac{\hbar}{\ell \mathbf{N}}. \quad (62)$$

As in Equation (61) at high  $|N_p|$ , ( $|N_p| \gg 1$ ),  $\Delta_p p_N = p_N - p_{N+1} \propto (\frac{1}{N_p} - \frac{1}{N_p+1}) = \frac{1}{N_p(N_p+1)}$ , it is clear that

$$N_p(N_p + 1) \leq \mathbf{N} \text{ or } -\frac{1}{2} - \sqrt{\frac{1}{4} + \mathbf{N}} \leq N_p \leq -\frac{1}{2} + \sqrt{\frac{1}{4} + \mathbf{N}}. \quad (63)$$

Considering that  $N_p$  is an integer number and  $\mathbf{N} \gg 1$ , it follows that

$$|N_p| \leq [\sqrt{\mathbf{N}}] - 1 \equiv \tilde{\mathbf{N}}, \quad (64)$$

where the square brackets [ ] in the right-hand side of Equation (64) denote an integer part of the number.

Next, due to Equations (60) and (61), an analog of Formulae (11) and (12) from [28] in the case under study at low energies will be of the form

$$\begin{aligned}\mathbf{p}.\psi(p) &\Rightarrow p_N\psi(p_N) = \frac{\hbar}{(N_p - \frac{1}{4N_p})\ell}\psi(p_N) \approx \frac{\hbar}{N_p\ell}\psi(p_N), \\ \mathbf{x}.\psi(p) &\Rightarrow \mathbf{x}.\psi(p_N) = i\hbar(1 + \frac{1}{(N_p - \frac{1}{4N_p})^2})\Delta_p^{-1}\psi(p_N) \approx \\ &\approx i\hbar(1 + \frac{1}{N_p^2})\Delta_p^{-1}\psi(p_N).\end{aligned}\quad (65)$$

The scalar product  $\langle\psi|\phi\rangle$  from [28]

$$\langle\psi|\phi\rangle = \int_{-\infty}^{+\infty} \frac{dp}{1 + \beta p^2} \psi^*(p)\phi(p) \quad (66)$$

in the case of low energies  $1 \ll |N_{\Delta p}| \leq \tilde{\mathbf{N}} < \infty$  is replaced by the sum

$$\begin{aligned}\langle\psi|\phi\rangle &= \int_{-\infty}^{+\infty} \frac{dp}{1 + \beta p^2} \psi^*(p)\phi(p) \Rightarrow \\ \Rightarrow \langle\psi|\phi\rangle_{1 \ll |N_p| \leq \tilde{\mathbf{N}}} &= \sum_{1 \ll |N_p| \leq \tilde{\mathbf{N}}} \frac{\Delta_p(p_N)\psi^*(p_N)\phi(p_N)}{(1 + \frac{1}{(N_p - \frac{1}{4N_p})^2})} \approx \\ &\approx \sum_{1 \ll |N_p| \leq \tilde{\mathbf{N}}} \frac{\Delta_p(p_N)\psi^*(p_N)\phi(p_N)}{(1 + \frac{1}{N_p^2})}.\end{aligned}\quad (67)$$

And since  $|N_p| \gg 1$  is a variable, in fact  $p_N$  is continuously varying and, proceeding from the above formulae, we can assume that to a high accuracy the function  $\phi(p_N), (\psi^*(p_N))$  is differentiable in terms of this variable.

On the other hand, at high energies, when for  $|N_p| \approx 1$  the presentation is fairly discrete, the scalar product Equation (66) is replaced by the sum

$$\begin{aligned}\langle\psi|\phi\rangle &= \int_{-\infty}^{+\infty} \frac{dp}{1 + \beta p^2} \psi^*(p)\phi(p) \Rightarrow \\ \Rightarrow \langle\psi|\phi\rangle_{|N_p| \approx 1} &= \sum_{|N_p| \approx 1} \frac{\Delta_p(p_N)\psi^*(p_N)\phi(p_N)}{(1 + \frac{1}{(N_p - \frac{1}{4N_p})^2})}.\end{aligned}\quad (68)$$

We consider only two cases: (a)  $1 \ll |N_p| \leq \tilde{\mathbf{N}}$ , “Quantum Consideration, Low Energies” and (b)  $|N_p| \approx 1$ , “Quantum Consideration, High Energies”. The case (c)

$$\tilde{\mathbf{N}} \ll |N_p| < \infty \quad (69)$$

is omitted in this Section as it is associated with the “Classical Picture”.

Then at all the energy scales  $\langle \psi | \phi \rangle_{N_p}$  may be formally represented as follows:

$$\langle \psi | \phi \rangle_{N_p} = \langle \psi | \phi \rangle_{1 \ll |N_p| \leq \tilde{\mathbf{N}}} + \langle \psi | \phi \rangle_{|N_p| \approx 1}. \quad (70)$$

However, with the formalism and terms proposed in this work, and also with the use of the Formula (12) that in this case takes the form

$$(|N_p| \approx 1) \rightarrow (1 \ll |N_p| \leq \tilde{\mathbf{N}}), \quad (71)$$

it seems more logical to consider the two components in Equation (70) separately, the first component originating in the process of the low-energy transition from the second component as follows:

$$\langle \psi | \phi \rangle_{|N_p| \approx 1} \xrightarrow{|N_p| \gg 1} \langle \psi | \phi \rangle_{1 \ll |N_p| \leq \tilde{\mathbf{N}}}. \quad (72)$$

Clearly, the first part of formula (13) from [28] holds as well in the general case for each of the components Equation (70)

$$\langle (\psi | \mathbf{p}) | \phi \rangle = \langle \psi | (\mathbf{p} | \phi) \rangle \quad (73)$$

The second part of formula (13) from [28]

$$\langle (\psi | \mathbf{x}) | \phi \rangle = \langle \psi | (\mathbf{x} | \phi) \rangle \quad (74)$$

takes place (to a high accuracy) for the low-energy case  $1 \ll |N_p| \leq \tilde{\mathbf{N}} < \infty$ , *i.e.*, for the first component in Equation (70).

Indeed, in this case, due to the condition  $|N_p| \gg 1$ , we have

$$\begin{aligned} \Delta_p p_N &\approx dp; \Delta_p^{-1} \psi(p_N) \approx \partial_p \psi(p_N) \\ &\text{or} \\ \lim_{|N_p| \rightarrow \infty, (\tilde{\mathbf{N}} \rightarrow \infty)} \Delta_p p_N &= dp; \lim_{|N_p| \rightarrow \infty, (\tilde{\mathbf{N}} \rightarrow \infty)} \Delta_p^{-1} \psi(p_N) = \partial_p \psi(p_N). \end{aligned} \quad (75)$$

Then in this (low-energy) case there exists the analog of formula (15) from [28]

$$\begin{aligned}
\langle \psi | (\mathbf{x} | \phi) \rangle &= \sum_{1 \ll |N_p| \leq \tilde{\mathbf{N}}-1} \frac{\Delta_p(p_N)}{(1 + \frac{1}{N_p^2})} \psi^*(p_N) i\hbar (1 + \frac{1}{N_p^2}) \Delta_p^{-1}(\phi(p_N)) = \\
&= \sum_{1 \ll |N_p| \leq \tilde{\mathbf{N}}-1} \Delta_p(p_N) \psi^*(p_N) i\hbar \Delta_p^{-1}(\phi(p_N)) \approx \\
&\approx \langle (\psi | \mathbf{x}) | \phi \rangle = \sum_{1 \ll |N_p| \leq \tilde{\mathbf{N}}-1} \Delta_p(p_N) (i\hbar \Delta_p^{-1} \psi(p_N))^* \phi(p_N). \quad (76)
\end{aligned}$$

It is important to note the following remarks:

- (1) The operator  $\mathbf{x}$  is defined in the case of low energies only for the functional space  $\phi(p_N)_{1 \ll |N_p| \leq \tilde{\mathbf{N}}-1}$ . Really, because of the existence of the Formula (61), the extreme point  $N_p$ , (such that  $(N_p + 1)(N_p + 2) > \mathbf{N}$ ) “moves” this operator beyond the domain under study  $\Delta p_{min} = p$ . Therefore, replacing  $N_p \mapsto N_p + 1$ ,  $N_p + 1 \mapsto N_p + 2$  in Formula (63), one can easily get the estimate of  $\tilde{\mathbf{N}} - 1$  instead of  $\mathbf{N}$  as seen in Equation (76).
- (2) Despite the fact that the operator  $\mathbf{x}$  is also defined at high energies, *i.e.*, for  $\phi(p_N)_{|N_p| \approx 1}$ , in general the property Equation (74) in this case has no place for lack of Formulae (75).
- (3) In all the cases when we consider  $|N_p| \gg 1$  (low energies) the “cut-off” for some upper bound  $p_{max}, (p_{max} \ll P_{pl}), 1 \ll N_{p_{max}} < |N_p|, p \neq p_{max}$  is determined by the initial conditions of the solved problem.
- (4) It is clear that in the relativistic case  $\Delta p_{min} = p$  leads to a minimal variation in the energy

$$|\Delta E_{min}| = (\Delta p)_{min} c = \frac{p}{\mathbf{N}} c. \quad (77)$$

- (5) In this work a minimal variation of the momentum  $\Delta p_{min}$  has been introduced from the additional assumptions but, as shown in [47], a minimal variation of the momentum may arise from the Extended Uncertainty

Principle (EUP) as follows:

$$\Delta x_i \Delta p_j \geq \hbar \delta_{ij} [1 + \beta^2 \frac{(\Delta x_i)^2}{l^2}], \quad (78)$$

where  $l$  is the characteristic, large length scale  $l \gg l_p$  and  $\beta$  is a dimensionless real constant on the order of unity [47]. From Equation (78) we get an absolute minimum in the momentum uncertainty

$$\Delta p_i \geq \frac{2\hbar\beta}{l}. \quad (79)$$

In [48] GUP and EUP are combined by the principle called the Symmetric Generalized Uncertainty Principle (SGUP):

$$\Delta x \Delta p \geq \hbar \left( 1 + \frac{(\Delta x)^2}{L^2} + l^2 \frac{(\Delta p)^2}{\hbar^2} \right), \quad (80)$$

where  $l \ll L$  and  $l$  defines the limit of the UV-cutoff (not being such up to a constant factor as in the case of GUP). Then

$$\Delta x_{\min} = 2l / \sqrt{1 - 4l^2/L^2} = \ell,$$

whereas  $L$  defines the limit for IR-cutoff *i.e.*, we have a

$$\Delta p_{\min} = 2\hbar / (L \sqrt{1 - 4l^2/L^2}).$$

(6) Of course, this paper is only the first step to resolve the Quantum Theory in terms of the measurable quantities using Definition 2. It is necessary to study thoroughly the low-energy case  $E \ll E_P$  and the correct transition to high energies  $E \propto E_P$ . The author is planning to treat these problems in his further works.

## 6 Gravity Markov's Model in Terms of Measurable Quantities

This heuristic model was introduced in the work [49] at the early eighties of the last century. This model already considered by the author in his

previous paper [46] is treated from the standpoint of the above-mentioned arguments. In [49], it is assumed that “by the universal decree of nature a quantity of the material density  $\varrho$  is always bounded by its upper value given by the expression that is composed of fundamental constants” ([49], p. 214):

$$\varrho \leq \varrho_p = \frac{c^5}{G^2 \hbar}, \quad (81)$$

with  $\varrho_p$  as “Planck’s density”.

Then the quantity

$$\wp_\varrho = \varrho / \varrho_p \leq 1 \quad (82)$$

is the deformation parameter as it is used in [49] to construct the following of Einstein equations deformation or  $\wp_\varrho$ -deformation (Formula (2) in [49]):

$$R_\mu^\nu - \frac{1}{2} R \delta_\mu^\nu = \frac{8\pi G}{c^4} T_\mu^\nu (1 - \wp_\varrho^2)^n - \Lambda \wp_\varrho^{2n} \delta_\mu^\nu, \quad (83)$$

where  $n \geq 1/2$ ,  $T_\mu^\nu$ —energy-momentum tensor,  $\Lambda$ —cosmological constant.

The case of the parameter  $\wp_\varrho \ll 1$  or  $\varrho \ll \varrho_p$  correlates with the classical Einstein equations, and the case when  $\wp_\varrho = 1$ —with the de Sitter Universe. In this way Equation (83) may be considered as  $\wp_\varrho$ -deformation of the General Relativity.

As shown in [46],  $\wp_\varrho$ -of Einstein equations deformation Equation (83) is nothing else but  $\alpha$ -deformation of GR for the parameter  $\alpha_l$  at  $a = l$  from Equation (13).

If  $\varrho = \varrho_l$  is the average material density for the Universe of the characteristic linear dimension  $l$ , *i.e.*, of the volume  $V \propto l^3$ , we have

$$\wp_{l,\varrho} = \frac{\varrho_l}{\varrho_p} \propto \alpha_l^2 = \omega \alpha_l^2, \quad (84)$$

where  $\omega$  is some computable factor.

Then it is clear that  $\alpha_l$ -representation Equation (83) is of the form

$$R_\mu^\nu - \frac{1}{2} R \delta_\mu^\nu = \frac{8\pi G}{c^4} T_\mu^\nu (1 - \omega^2 \alpha_l^4)^n - \Lambda \omega^{2n} \alpha_l^{4n} \delta_\mu^\nu, \quad (85)$$

or in the general form we have

$$R_\mu^\nu - \frac{1}{2} R \delta_\mu^\nu = \frac{8\pi G}{c^4} T_\mu^\nu(\alpha_l) - \Lambda(\alpha_l) \delta_\mu^\nu. \quad (86)$$

But, as r.h.s. of Equation (86) is dependent on  $\alpha_l$  of any value and particularly in the case  $\alpha_l \ll 1$ , *i.e.*, at  $l \gg \ell$ , l.h.s of Equation (86) is also dependent on  $\alpha_l$  of any value and Equation (86) may be written as

$$R_\mu^\nu(\alpha_l) - \frac{1}{2}R(\alpha_l)\delta_\mu^\nu = \frac{8\pi G}{c^4}T_\mu^\nu(\alpha_l) - \Lambda(\alpha_l)\delta_\mu^\nu. \quad (87)$$

Thus, in this specific case we obtain the explicit dependence of GR on the available energies  $E \sim \frac{1}{l}$ , that is insignificant at low energies or for  $l \gg \ell$  and, on the contrary, significant at high energies,  $l \rightarrow \ell$ .

## 6.1 Low Energies, $E \ll E_P$

**1.** Low energies. Nonmeasurable case. In this case at low energies, using Formula (13) in the limit  $\ell = 0$  for  $a = l$ , we get a continuous theory coincident with the General Relativity.

**2.** Low energies. Measurable case. In this case at low energies, using Formulas (13) and (18) for  $\ell \neq 0$ , for  $a = l$  (and hence for  $N_l \gg 1$ ), we get a discrete theory which is a “nearly continuous theory”, practically similar to the General Relativity with the slowly (smoothly) varying parameter  $\alpha_{l(t)}$ , where  $t$ —time.

So, due to low energies and momentums ( $E \ll E_P, p \ll P_{Pl}$ ), the “continuous case” **1** (General Relativity) and the “discrete case” **2** that is actually a “nearly continuous case”.

## 6.2 High Energies, $E \approx E_P$

At high energies we consider the measurable case only. Then it is clear that at high energies the parameter  $\alpha_{l(t)}$  is discrete and for the limiting value of  $\alpha_{l(t)} = 1$  we get a discrete series of equations of the form Equation (87) (or a single equation of this form met by a discrete series of solutions) corresponding to  $\alpha_{l(t)} = 1; 1/4; 1/9; \dots$

As this takes place,  $T_\mu^\nu(\alpha_l) \approx 0$ , and in both cases as **2** in 6.1 as well as 6.2  $\Lambda(\alpha_l)$  is not longer a cosmological constant, being a dynamical cosmological term.



Note that because of Formula (20) given in Section 3,  $\sqrt{\alpha_{l(t)}}$  in cases **2** in 6.1 and 6.2 is an element of the lattice  $Lat_{P-E}$  from Section 4. And in case **2** it is an element of the sublattice  $Lat_{P-E}[LE]$ , whereas case 6.2 is associated with the sublattice  $Lat_{P-E}[HE]$ .

It seems expedient to make some important remarks:

(1) In formulae (71) and (72) of Section 5 in this work we have considered the transition

$$\begin{aligned} & \textit{Quantum Theory in High Energies (QTHE)} \Rightarrow \\ & \Rightarrow \textit{Quantum Theory in Low Energies (QTLE)}. \end{aligned} \quad (88)$$

However, according to the modern knowledge, the (quantum) gravity phase begins only at very high energies at Planck scales, *i.e.*, the case (a) from Section 5 is inexistent, and hence it is more correct to consider the transition

$$\begin{aligned} & \textit{Quantum Theory in High Energies (QTHE)} \Rightarrow \\ & \Rightarrow \textit{Classical Theory (Low Energies)}. \end{aligned} \quad (89)$$

And this corresponds to the case (c) that has been omitted from consideration in Section 5 Equation (69) with  $\tilde{\mathbf{N}} = 1$

$$(|N_p| \approx 1) \rightarrow (1 \ll |N_p| < \infty). \quad (90)$$

(2) Generally speaking, as 6.2 and case **2** in 6.1 are associated with measurable cases for low energies and high energies, respectively, all the terms of the Equation (87):  $R_\mu^\nu(\alpha_l), R(\alpha_l), T_\mu^\nu(\alpha_l), \Lambda(\alpha_l)$  must be expressed in terms of measurable quantities in view of Definition 2 from Section 3. But this problem still remains to be solved. In fact, it is reduced to the construction of the following “*measurable*” deformations in the sense of Definition 2 in Section 3 as follows:

$$\begin{aligned} \lim_{\ell \rightarrow 0} (R_\mu^\nu(\alpha_l \ll 1), R(\alpha_l \ll 1), T_\mu^\nu(\alpha_l \ll 1), \Lambda(\alpha_l \ll 1)) \rightarrow \\ \rightarrow (R_\mu^\nu, R, T_\mu^\nu, \Lambda) \end{aligned} \quad (91)$$

and

$$\begin{aligned}
& \lim_{(\alpha_l \approx 1) \rightarrow (\alpha_l \ll 1)} (R_\mu^\nu(\alpha_l \approx 1), R(\alpha_l \approx 1), T_\mu^\nu(\alpha_l \approx 1), \Lambda(\alpha_l \approx 1)) \rightarrow \\
& \rightarrow \lim_{l_{min} \rightarrow 0} (R_\mu^\nu(\alpha_l \ll 1), R(\alpha_l \ll 1) \delta_\mu^\nu, T_\mu^\nu(\alpha_l \ll 1), \Lambda(\alpha_l \ll 1)) \rightarrow \\
& \rightarrow (R_\mu^\nu, R, T_\mu^\nu, \Lambda). \quad (92)
\end{aligned}$$

Here the first Equation (91) is a pure low-energy limiting transition from the measurable variant of gravity to the nonmeasurable one (or from a discrete theory to a continuous theory), whereas the second Equation (92) from the beginning is associated with the measurable transition from high energies to low energies and then is coincident with the first one.

**(3)** It should be noted that in [1, 2] in terms of measurable quantities, as an example, we have studied gravity for the static spherically-symmetric horizon space. It has been shown that, “...despite the absence of infinitesimal spatial-temporal increments owing to the existence of  $l_{min}$  and the essential ‘discreteness’ of a theory, this discreteness at low energies is not ‘felt’, the theory in fact being close to the original continuum theory. The indicated discreteness is significant only in the case of high (Planck) energies ” [1]. The Markov model considered in this section represents the generalization of the above-mentioned example. Of course, this model requires further thorough investigation in terms of measurable quantities.

## 7 Conclusions

### 7.1 Measurable and Non-Measurable Transitions in Gravity

The illustration considered in the preceding Section (Gravity Markov’s Model) is universal considering the following:

First, using the formalism of this work, it is required to construct a measurable deformation of the General Relativity (GR) at low energies (Formula (91)). This deformation is denoted in terms of  $Grav[LE]^\ell$

$$Grav[LE]^\ell \xrightarrow{\ell \rightarrow 0} GR. \quad (93)$$

Next, we should construct the high-energy deformation (denoted in terms of  $Grav[HE]^\ell$ ), this time for  $Grav[LE]^\ell$  (the first arrow in the Formula (92))

$$Grav[HE]^\ell \xrightarrow{\alpha_l \rightarrow 0} Grav[LE]^\ell. \quad (94)$$

At the present time the majority of the proposed approaches to quantization of gravity are associated with the construction of the following transition:

$$GR \Rightarrow Grav[HE]^\ell. \quad (95)$$

But, by the author opinion, this is impossible. It seems that for correct quantization of gravity one needs reversal of the arrow from Equation (94)

$$Grav[LE]^\ell(\alpha_l \approx 0, \alpha_l \neq 0) \xrightarrow{\alpha_l \rightarrow 1} Grav[HE]^\ell(\alpha_l \approx 1). \quad (96)$$

The above results indicate that the low-energy “measurable” gravity variant  $Grav[LE]^\ell$  should be very close to GR but different at the same time.

The author is hopeful that the correct construction of a low-energy  $Grav^\ell$  close to GR allows for a more natural transition to quantum (Planck) gravity. Besides, within the notion of measurability, gravity could be saved from some odd solutions, from wormholes in particular.

## 7.2 Measurable and Non-measurable Transitions in Quantum Theory

The situation is similar for a quantum theory too. In the general case, based on the parameter  $\alpha_a$  (Formula (18) of Section 3), this means that there exists the following correct limiting high-energy transition:

$$\lim_{\ell \neq 0, |N_a| \gg 1} \alpha_a \xrightarrow{High \Rightarrow Energy} \lim_{\ell \neq 0, |N_a| \approx 1} \alpha_a \quad (97)$$

and there is no correct limiting high-energy transition

$$\lim_{\ell=0} \alpha_a \xrightarrow{High \Rightarrow Energy} \lim_{\ell \neq 0, |N_a| \approx 1} \alpha_a. \quad (98)$$

The first of them corresponds to the transition from a measurable theory at low energies to a measurable theory at high energies

$$QT[LE]^\ell \xrightarrow{N_a \rightarrow 1} QT[HE]^\ell. \quad (99)$$

Whereas the second

$$QT \xrightarrow{N_a \rightarrow 1} QT[HE]^\ell \quad (100)$$

(here  $QT[LE]^\ell$ ,  $QT[HE]^\ell$ ,  $QT$  are quantum theories with the minimal length  $\ell \neq 0$  at low energies  $E \ll E_p$ , at high energies  $E \approx E_p$ , and the well-known (continuous) quantum theory with  $l_{min} = 0$ ).

However, the whole theoretical physics, where presently at low energies  $E \ll E_p$  the minimal length  $\ell$  is not involved at all (*i.e.*,  $l_{min} = 0$ ), is framed around a search for the nonexistent limits Equation (98) (correspondingly Equation (100)).

Of course, in this case the low-energy “measurable” variant  $QT[LE]^\ell$  of  $QT$  by its results will be very close to the initial theory  $QT$ , as indicated in [1, 2], and Section 5 of the present work. But these theories are different by nature: the first of them is discrete and the second one is continuous. Nevertheless, it is clear that the main requirement in this case is associated with the “Compatibility Principe”:

*at low energies the resolved variant  $QT[LE]^\ell$  must, to a high accuracy, represent the well-known approved results of the corresponding continuous theory  $QT$ .*

These theories should be differing considerably at least on going to high energies  $E \approx E_p$ .

The hypothesis set by the author is that correct construction of the “measurable” transition to high energies (Formula (99)) should naturally lead to solution of the ultraviolet divergences problem (initially in terms of the finite measurable quantities).

### 7.3 Summary of 7.1 and 7.2 is Such [2]

1. When in the theory the minimal length  $l_{min} \neq 0$  is actualized (involved) at all the energy scales, a mathematical apparatus of this theory must be

changed considerably: no infinitesimal space-time variations (increments) must be involved, the key role being played by the definition of measurability (Definition 2 from Section 3).

**2.** As this takes place the theory becomes discrete at all the energy scales but at low energies (far from the Planck energies) the sought for theory must be very close in its results to the starting continuous theory (with  $l_{min} = 0$ ). In the process a real discreteness is exhibited only at high energies which are close to the Planck energies.

**3.** By this approach the theory at low and high energies is associated with a common single set of the parameters ( $N_L$  from Formula (10)) or with the dimensionless small parameters ( $1/N_L = \sqrt{\alpha_L}$ ) which are lacking if at low energies the theory is continuous, *i.e.*, when  $l_{min} = 0$ .

The principal objective of my further studies is to develop for quantum theory and gravity, within the scope of the considerations given in points **1–3**, the corresponding discrete models (with  $l_{min} \neq 0$ ) for all the energy scales and to meet the following requirements:

**4.** At low energies the models must, to a high accuracy, represent the results of the corresponding continuous theories.

**5.** The models should not have the problems of transition from low to high energies and, specifically, the ultraviolet divergences problem. By author's opinion, the problem associated with points **4** and **5** is as follows.

**6.** It is interesting to know why, with the existing  $l_{min} \neq 0, t_{min} \neq 0$  and discreteness of nature, at low energies  $E \ll E_{max} \propto E_P$  the apparatus of mathematical analysis based on the use of infinitesimal space-time quantities ( $dx_\mu, \frac{\partial \varphi}{\partial x_\mu}$ , and so on) is very efficient giving excellent results. The answer is simple: in this case  $l_{min}$  and  $t_{min}$  are very far from the available scale of  $L$  and  $t$ , the corresponding  $N_L \gg 1, N_t \gg 1$  being in general true but insufficient. There is a need for rigorous calculations.

### Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this article.

## References

- [1] Shalyt-Margolin, A.E. Minimal Length and the Existence of Some Infinitesimal Quantities in Quantum Theory and Gravity. *Adv. High Energy Phys.* **2014**, 2014, 8, doi:10.1155/2014/195157.
- [2] Shalyt-Margolin, A.E. Holographic Principle, Minimal Length and Measurability. *J. Adv. Phys.* (In press)
- [3] Wald ,Robert. M. *General Relativity*; Chicago:The University Chicago Press,USA,1984.
- [4] Amelino-Camelia, G. Quantum Spacetime Phenomenology. *Living Rev. Relativ.* **2013**, 16, 5–129.
- [5] Penrose, R. *Quantum Theory and Space-Time*, Fourth Lecture in *Stephen Hawking and Roger Penrose, The Nature of Space and Time*; Prinseton University Press: Princeton, NJ, USA, 1996.
- [6] Garay, L. Quantum gravity and minimum length. *Int. J. Mod. Phys. A* **1995**, 10, 145–146.
- [7] Amelino-Camelia, G.; Smolin, L. Prospects for constraining quantum gravity dispersion with near term observations. *Phys. Rev. D* **2009**, doi:10.1103/PhysRevD.80.084017.
- [8] Gubitosi, G.; Pagano, L.; Amelino-Camelia, G.; Melchiorri, A.; Cooray, A. A constraint on planck-scale modifications to electrodynamics with CMB polarization data. *J. Cosmol. Astropart. Phys.* **2009**, 908, 21–34.
- [9] Amelino-Camelia, G. Building a case for a planck-scale-deformed boost action: The planck-scale particle-localization limit. *Int. J. Mod. Phys. D* **2005**, 14, 2167–2180.
- [10] Hossenfelder, S.; Bleicher, M.; Hofmann, S.; Ruppert, J.; Scherer, S.; Stöcker, H. Signatures in the Planck Regime. *Phys. Lett. B* **2003**, 575, 85–99.

- [11] Hossenfelder, S. Running Coupling with Minimal Length. *Phys. Rev. D* **2004**, doi:10.1103/PhysRevD.70.105003.
- [12] Hossenfelder, S. Self-consistency in Theories with a Minimal Length. *Class. Quantum Gravity* **2006**, *23*, 1815–1821.
- [13] Hossenfelder, S. Minimal Length Scale Scenarios for Quantum Gravity. *Living Rev. Relativ.* **2013**, doi:10.12942/lrr-2013-2.
- [14] Heisenberg, W. Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. *Z. Phys.* **1927**, *43*, 172–198. (In German)
- [15] Messiah, A. *Quantum Mechanics*; North Holland Publishing Company: Amsterdam, The Netherlands, 1967; Volume 1.
- [16] Berestetskii, V.B.; Lifshitz, E.M.; Pitaevskii, L.P. *Relativistic Quantum Theory*; Pergamon: Oxford, UK, 1971.
- [17] Veneziano, G.A. Stringy nature needs just two constants. *Europhys. Lett.* **1986**, *2*, 199–211.
- [18] Amati, D.; Ciafaloni, M.; Veneziano, G. Can spacetime be probed below the string size? *Phys. Lett. B* **1989**, *216*, 41–47.
- [19] Witten, E. Reflections on the fate of spacetime. *Phys. Today* **1996**, *49*, 24–28.
- [20] Polchinski, J. *String Theory*; Cambridge University Press: Cambridge, UK, 1998.
- [21] Adler, R.J.; Santiago, D.I. On gravity and the uncertainty principle. *Mod. Phys. Lett. A* **1999**, *14*, 1371–1378.
- [22] Ahluwalia, D.V. Wave-particle duality at the Planck scale: Freezing of neutrino oscillations. *Phys. Lett. A* **2000**, *275*, 31–35.
- [23] Ahluwalia, D.V. Interface of gravitational and quantum realms. *Mod. Phys. Lett. A* **2002**, *17*, 1135–1145.

- [24] Maggiore, M. The algebraic structure of the generalized uncertainty principle. *Phys. Lett. B* **1993**, *319*, 83–86.
- [25] Maggiore, M. Black Hole Complementarity and the Physical Origin of the Stretched Horizon. *Phys. Rev. D* **1994**, *49*, 2918–2921.
- [26] Maggiore, M. A Generalized Uncertainty Principle in Quantum Gravity. *Phys. Rev. D* **1993**, doi:10.1016/0370-2693(93)91401-8.
- [27] Capozziello, S.; Lambiase, G.; Scarpetta, G. The Generalized Uncertainty Principle from Quantum Geometry. *Int. J. Theor. Phys.* **2000**, *39*, 15–22.
- [28] Kempf, A.; Mangano, G.; Mann, R.B. Hilbert space representation of the minimal length uncertainty relation. *Phys. Rev. D* **1995**, *52*, 1108–1118.
- [29] Nozari, K.; Etemadi, A. Minimal length, maximal momentum and Hilbert space representation of quantum mechanics. *Phys. Rev. D* **2012**, doi:10.1103/PhysRevD.85.104029.
- [30] Shalyt-Margolin, A.E.; Suarez, J.G. Quantum Mechanics of the Early Universe and Its Limiting Transition. Available online: <http://arxiv.org/abs/gr-qc/0302119> (accessed on 30 August 2003).
- [31] Shalyt-Margolin, A.E.; Suarez, J.G. Quantum mechanics at Planck scale and density matrix. *Int. J. Mod. Phys. D* **2003**, *12*, 1265–1278.
- [32] Shalyt-Margolin, A.E.; Tregubovich, A.Y. Deformed density matrix and generalized uncertainty relation in thermodynamics. *Mod. Phys. Lett. A* **2004**, *19*, 71–82.
- [33] Shalyt-Margolin, A.E. Non-unitary and unitary transitions in generalized quantum mechanics, new small parameter and information problemsolving. *Mod. Phys. Lett. A* **2004**, *19*, 391–403.
- [34] Shalyt-Margolin, A.E. Pure states, mixed states and Hawking problem in generalized quantum mechanics. *Mod. Phys. Lett. A* **2004**, *19*, 2037–2045.



- [35] Shalyt-Margolin, A.E. The universe as a nonuniform lattice in finite-volume hypercube: I. Fundamental definitions and particular features. *Int. J. Mod. Phys. D* **2004**, *13*, 853–864.
- [36] Shalyt-Margolin, A.E. The Universe as a nonuniform lattice in the finite-dimensional hypercube. II. Simple cases of symmetry breakdown and restoration. *Int. J. Mod. Phys. A* **2005**, *20*, 4951–4964.
- [37] Shalyt-Margolin, A.E. The density matrix deformation in physics of the early universe and some of its implications. In *Quantum Cosmology Research Trends*; Reimer, A., Ed.; Nova Science: Hauppauge, NY, USA, 2005; pp. 49–92.
- [38] Faddeev, L. Mathematical view of the evolution of physics. *Priroda* **1989**, *5*, 11–16.
- [39] Landau, L.D.; Lifshits, E.M. *Field Theory*; Theoretical Physics: Moscow, Russia, 1988; Volume 2.
- [40] Hossenfelder, S. Minimal Length Scale Scenarios for Quantum Gravity. *Living Rev. Relativ.* **2013**, *36*, 2, doi:10.12942/lrr-2013-2.
- [41] Tawfik, A.N.; Diab, A.M. Generalized Uncertainty Principle: Approaches and Applications. *Int. J. Mod. Phys. D* **2014**, *23*, 1430025.
- [42] Vagenas, E.C.; Majhi, B.R. Modified Dispersion Relation, Photon’s Velocity, and Unruh Effect. *Phys. Lett. B* **2013**, *725*, 477–483.
- [43] Nozari, K.; Sefiedgar, A.S. Comparison of Approaches to Quantum Correction of Black Hole Thermodynamics. *Phys. Lett. B* **2006**, *635*, 156–160.
- [44] Nozari, K.; Fazlpour, B. Generalized Uncertainty Principle, Modified Dispersion Relations and Early Universe Thermodynamics. *Gen. Relativ. Gravit.* **2006**, *38*, 1661–1679.
- [45] Shalyt-Margolin, A. Entropy in the present and early universe: New small parameters and dark energy problem. *Entropy* **2010**, *12*, 932–952.

- [46] Shalyt-Margolin, A.E. Quantum theory at planck scale, limiting values, deformed gravity and dark energy problem. *Int. J. Mod. Phys. D* **2012**, *21*, doi:10.1142/S0218271812500137.
- [47] Park, M.I. The Generalized Uncertainty Principle in (A)dS Space and the Modification of Hawking Temperature from the Minimal Length. *Phys. Lett. B* **2008**, *659*, 698–702.
- [48] Kim, W.; Son, E.J.; Yoon, M. Thermodynamics of a black hole based on a generalized uncertainty principle. *JHEP* **2008**, *8*, doi:10.1088/1126-6708/2008/01/035.
- [49] Markov, M.A. Ultimate Matter Density as the Universal Low of Nature. *JETP Lett.* **1982**, *36*, 214–216.